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ON THE SOLUTION OF THE NONLINEAR HEAT CONDUCTION EQUATIONS BY NUMERICAL METHODS

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Abstract—A three-time level implicit scheme for the numerical solution of the multidimensional heat conduction equations when the thermophysical properties depend on temperature is presented. As the thermophysical properties are evaluated at the intermediate time level, the complication of solving a set of nonlinear equations at each time step is avoided. In the case of boundary conditions of the first kind the method is shown to be unconditionally stable and convergent. Boundary conditions of the second and third kind are then dealt with in a similar way by suitably defining "equivalent thermal conductivities".

This procedure is used to simulate several convective heating and cooling tests on "Tylose" samples which have strongly temperature dependent thermophysical properties. Temperature fields in squares subjected to boundary conditions of the second and third kind are thus computed and satisfactorily compared with the results of the experiments.

NOMENCLATURE

A, B, negative constants;

c, specific heat [J/kg K];

C, volumetric heat capacity [J/m^3 K];

I, imaginary unit;

k, thermal conductivity [W/m K];

$0(\Delta x^6)$, the remainder in Taylor series expansion and of the order (Δx^6);

q, heat flux [J/s];

t, temperature: solution of the heat conduction equations [°C];

T, temperature: solution of the numerical equations [°C];

x, y, position coordinates [m];

X, Y, difference operators defined by equation (15).

Greek letters

α, convective heat transfer coefficient [W/m^2 K];

γ_x, γ_y, positive constants [m^{-1}];

δ_x, difference operator: $\delta_x T_{i,j} = T_{i,j+\frac{1}{2}} - T_{i,j-\frac{1}{2}}$;

δ_y, difference operator: $\delta_y T_{i,j} = T_{i+\frac{1}{2},j} - T_{i-\frac{1}{2},j}$;

ΔL, grid spacing [m];

$\Delta x, \Delta y$, grid spacing in the x, y direction [m];

$\Delta\tau$, time step [s];

ε, local order of accuracy [K/s];

$[\phi_n]$, two-dimensional column vector, function of time in the Fourier expansion;

λ, root of the characteristic equation (19);

θ, angle of inclination: cf. Fig. 1b;

$(\mu\delta)_\tau$, difference operator:

$$(\mu\delta)_\tau T^h = \frac{\delta_\tau T^{h+\frac{1}{2}} + \delta_\tau T^{h-\frac{1}{2}}}{2} = \frac{T^{h+1} - T^{h-1}}{2};$$

ρ, density [kg/m^3];

τ, time [s].

Subscripts

e, external;

eq, equivalent;

i, j, lattice parameters in the two-dimensional grid, $x = i\Delta x$, $y = j\Delta y$;

x, y, τ, in direction of x, y, τ.

Superscripts

h, time level, $\tau = h\Delta\tau$.

INTRODUCTION

SEVERAL finite difference methods have been suggested for the solution of the unsteady state heat conduction equations even for problems involving complex geometries, non homogeneous materials and time dependent boundary conditions [5, 11, 12]. However, when thermophysical properties vary with temperature, the coefficients of the finite difference equations may vary from one time step to the next. They must therefore be first evaluated at a suitably chosen average temperature so as to achieve afterwards, by means of successive iterations, a reasonable accuracy. Under such circumstances, it does appear that difference methods based on two-time levels are difficult to use computationally. In fact when explicit methods are employed the time step has to be strictly limited and the limitation depends on the values of the thermophysical properties. On the other hand, when implicit methods are tried, the resulting system of algebraic equations to be solved at each time step is nonlinear. Accordingly, a three-level method has been considered here as this allows a direct evaluation of thermophysical properties at the intermediate time level, thus eliminating the need of subsequent iterations. Such a procedure has been suggested by Lees for the numerical integration of quasilinear parabolic equations with one-dimensional geometries and boundary conditions of the first kind [9]. The Lees method has already been successfully used by the authors for the solution of one-dimensional conduction and diffusion problems, in the case of material parameters dependent on temperature and/or specific humidity [1, 3]. In this paper the extension of the scheme to multi-dimensional geometries and to boundary conditions of the second and third kind is considered. The accuracy of the procedure thus developed is demonstrated by comparing the results of the numerical calculations with those of several convective heating and cooling tests. The test material used in the experiments is "Tylose", a substance whose strongly temperature dependent thermophysical properties are about the same as those of lean beef [13]. The information thus obtained can therefore be immediately applied to forecasting the thermal behaviour of foodstuffs under refrigeration. Possible uses of the technique herein described concern also other applications of considerable scientific and technical interest such as the determination of thermal fields in materials at cryogenic temperatures [6] and the calculation of mass losses in foodstuffs during refrigeration and cold storage [2].

PROBLEM FORMULATION AND NUMERICAL METHOD

The problem which is considered here is the numerical solution of the quasilinear parabolic equation typical of unsteady thermal fields in substances with thermophysical properties dependent on temperature:

$$\rho c \frac{\partial t}{\partial \tau} = \operatorname{div}(k \operatorname{grad} t). \tag{1}$$

For the sake of simplicity a two-dimensional configuration will be examined; the numerical scheme of the three-dimensional case however is analogous, even though formally more complicated, and is described in detail in [4]. One-dimensional problems having been already dealt with elsewhere [3] will not be treated here.

The derivation of a finite-difference formula for the two-dimensional case is a straightforward extension of the Lees procedure. Referring to the general equation (1) and to Fig. 1a the first differentials of temperature with respect to time and space coordinates are approximated by means of the operators $\mu\delta$ and δ respectively, while on the right side of the difference equation the average temperature:

$$\hat{T}^h_{i,j} = \tfrac{1}{3}(T^{h+1}_{i,j} + T^h_{i,j} + T^{h-1}_{i,j})$$

is used in place of $T^h_{i,j}$.

The following difference equation is obtained:

$$\frac{C(T^h_{i,j})}{\Delta\tau}(\mu\delta)_\tau T^h_{i,j} = \frac{1}{\Delta x^2}\delta_x[k(T^h_{i,j})\delta_x \hat{T}^h_{i,j}] + \frac{1}{\Delta y^2}\delta_y[k(T^h_{i,j})\delta_y \hat{T}^h_{i,j}]$$

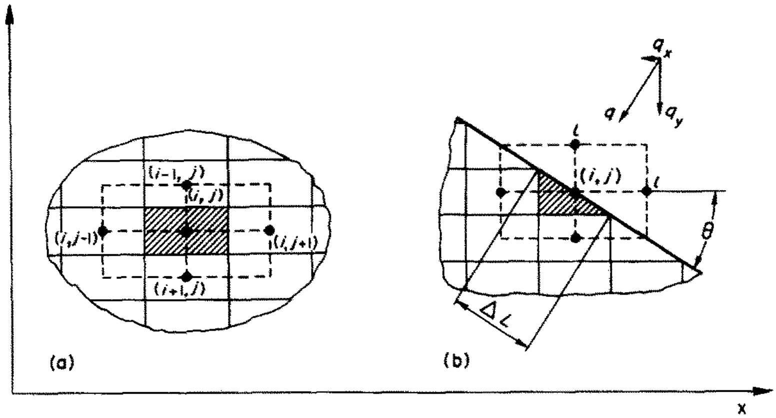

FIG. 1. Discrete elements in a spatial mesh. (a) Internal point; (b) surface point with a diagonal boundary.

which can be rewritten avoiding to fully indicate the functional dependences of k and C on temperature so as to reduce the notations:

$$\left[1-\frac{2}{3}\frac{\Delta\tau}{\Delta x^2 C}\delta_x(k\delta_x)-\frac{2}{3}\frac{\Delta\tau}{\Delta y^2 C}\delta_y(k\delta_y)\right]\times T^{h+1}_{i,j}=\left[\frac{2}{3}\frac{\Delta\tau}{\Delta x^2 C}\delta_x(k\delta_x)+\frac{2}{3}\frac{\Delta\tau}{\Delta y^2 C}\delta_y(k\delta_y)\right]\times T^{h}_{i,j}+\left[1+\frac{2}{3}\frac{\Delta\tau}{\Delta x^2 C}\delta_x(k\delta_x)+\frac{2}{3}\frac{\Delta\tau}{\Delta y^2 C}\times\delta_y(k\delta_y)\right]T^{h-1}_{i,j}.$$

In order to obtain an operator on the left side which factorizes, the fourth order terms

$$\frac{4}{9}\frac{\Delta\tau^2}{C^2\,\Delta x^2\,\Delta y^2}\left[\delta_x(k\delta_x)\delta_y(k\delta_y)\right]T^{h+1}_{i,j}$$

and

$$\frac{4}{9}\frac{\Delta\tau^2}{C^2\,\Delta x^2\,\Delta y^2}\left[\delta_x(k\delta_x)\delta_y(k\delta_y)\right]T^{h-1}_{i,j}$$

are added respectively to the left and right side, leading to:

$$\left[1-\frac{2}{3}\frac{\Delta\tau}{C\,\Delta x^2}\delta_x(k\delta_x)\right]\left[1-\frac{2}{3}\frac{\Delta\tau}{C\,\Delta y^2}\delta_y(k\delta_y)\right]\times T^{h+1}_{i,j}=\left[\frac{2}{3}\frac{\Delta\tau}{C\,\Delta x^2}\delta_x(k\delta_x)+\frac{2}{3}\frac{\Delta\tau}{C\,\Delta y^2}\times\delta_y(k\delta_y)\right]T^{h}_{i,j}+\left[1+\frac{2}{3}\frac{\Delta\tau}{C\,\Delta x^2}\delta_x(k\delta_x)\right]\times\left[1+\frac{2}{3}\frac{\Delta\tau}{C\,\Delta y^2}\delta_y(k\delta_y)\right]T^{h-1}_{i,j} \qquad (2)$$

where the fourth order terms added do not alter the order of accuracy but considerably simplify the structure of the derived formulae.

Equation (2) can be split by introducing a temperature intermediate value $T^{(h+1)*}_{i,j}$ into the two formulae:

$$\left[1-\frac{2}{3}\frac{\Delta\tau}{C\,\Delta x^2}\delta_x(k\delta_x)\right]T^{(h+1)*}_{i,j}=\left[\frac{2}{3}\frac{\Delta\tau}{C\,\Delta x^2}\times\delta_x(k\delta_x)+\frac{2}{3}\frac{\Delta\tau}{C\,\Delta y^2}\delta_y(k\delta_y)\right]T^{h}_{i,j}+\left[1+\frac{2}{3}\frac{\Delta\tau}{C\,\Delta x^2}\delta_x(k\delta_x)+\frac{4}{3}\frac{\Delta\tau}{C\,\Delta y^2}\times\delta_y(k\delta_y)\right]T^{h-1}_{i,j} \qquad (3')$$

and

$$\left[1-\frac{2}{3}\frac{\Delta\tau}{C\,\Delta y^2}\delta_y(k\delta_y)\right]T^{h+1}_{i,j}=T^{(h+1)*}_{i,j}-\frac{2}{3}\frac{\Delta\tau}{C\Delta y^2}\times\delta_y(k\delta_y)\,T^{h-1}_{i,j}. \qquad (3'')$$

The split formulae (3′) and (3″) involve the solution of tridiagonal sets of equations along rows and columns respectively at the first and second half-step; therefore they represent an alternating direction implicit method.

In a form more suitable for coding in a symbolic language like, for instance, the Fortran one, (3′) and (3″) can be written as:

$$-\frac{2}{3}\frac{\Delta\tau}{C\,\Delta x^2}k_x^- T_{i,j-1}^{(h+1)*} + \left[1 + \frac{2}{3}\frac{\Delta\tau}{C\,\Delta x^2} \times (k_x^+ + k_x^-)\right] T_{i,j}^{(h+1)*} - \frac{2}{3}\frac{\Delta\tau}{C\,\Delta x^2}k_x^+ T_{i,j+1}^{(h+1)*}$$
$$= \left\{\frac{2}{3}\frac{\Delta\tau}{C\,\Delta x^2}\left[k_x^+(T_{i,j+1}^h - T_{i,j}^h) - k_x^-(T_{i,j}^h - T_{i,j-1}^h)\right] + \frac{2}{3}\frac{\Delta\tau}{C\,\Delta y^2} \times \left[k_y^+(T_{i+1,j}^h - T_{i,j}^h) - k_y^-(T_{i,j}^h - T_{i-1,j}^h)\right]\right\}$$
$$+ \left\{T_{i,j}^{h-1} + \frac{2}{3}\frac{\Delta\tau}{C\,\Delta x^2}\left[k_x^+(T_{i,j+1}^{h-1} - T_{i,j}^{h-1}) - k_x^-(T_{i,j}^{h-1} - T_{i,j-1}^{h-1})\right] + \frac{4}{3}\frac{\Delta\tau}{C\,\Delta y^2}\left[k_y^+ \times (T_{i+1,j}^{h-1} - T_{i,j}^{h-1}) - k_y^-(T_{i,j}^{h-1} - T_{i-1,j}^{h-1})\right]\right\} \quad (4')$$

and:

$$-\frac{2}{3}\frac{\Delta\tau}{C\,\Delta y^2}k_y^- T_{i-1,j}^{h+1} + \left[1 + \frac{2}{3}\frac{\Delta\tau}{C\,\Delta y_2}(k_y^+ + k_y^-)\right] \times T_{i,j}^{h+1} - \frac{2}{3}\frac{\Delta\tau}{C\,\Delta y^2}k_y^+ T_{i+1,j}^{h+1}$$
$$= T_{i,j}^{(h+1)*} - \frac{2}{3}\frac{\Delta\tau}{C\,\Delta y^2}\left[k_y^+(T_{i+1,j}^{h-1} - T_{i,j}^{h-1}) - k_y^-(T_{i,j}^{h-1} - T_{i-1,j}^{h-1})\right] \quad (4'')$$

where:

$$C = C(T_{i,j}^h) \quad (5)$$

$$k_x^+ = k(T_{i,j+\frac{1}{2}}^h) \cong k\left(\frac{T_{i,j+1}^h + T_{i,j}^h}{2}\right) \quad (6)$$

$$k_x^- = k(T_{i,j-\frac{1}{2}}^h) \cong k\left(\frac{T_{i,j}^h + T_{i,j-1}^h}{2}\right) \quad (7)$$

$$k_y^+ = k(T_{i+\frac{1}{2},j}^h) \cong k\left(\frac{T_{i+1,j}^h + T_{i,j}^h}{2}\right) \quad (8)$$

$$k_y^- = k(T_{i-\frac{1}{2},j}^h) \cong k\left(\frac{T_{i,j}^h + T_{i-1,j}^h}{2}\right). \quad (9)$$

These latter expressions too do not alter the order of accuracy and, moreover, involve values of temperature at grid points only. Another noteworthy feature of formulae (4′) and (4″) is that the two sets of equations to be solved at each time step are linear since the thermophysical properties are evaluated at the intermediate time level: resorting to iterations within each time interval is thus avoided.

LOCAL ACCURACY, STABILITY AND CONVERGENCE

The local order of accuracy of the present method, defined as the difference between the finite difference formula (2) and the differential equation, is obtained from a Taylor's series expansion of the function $t(x, y, \tau)$ by using the relationships:

$$\delta_x(k\delta_x) = \Delta x^2\frac{\partial}{\partial x}\left(k\frac{\partial}{\partial x}\right) + \frac{\Delta x^4}{24}\left[\frac{\partial}{\partial x}\left(k\frac{\partial^3}{\partial x^3}\right) + \frac{\partial^3}{\partial x^3}\left(k\frac{\partial}{\partial x}\right)\right] + 0(\Delta x^6), \quad (10)$$

$$\delta_y(k\delta_y) = \Delta y^2\frac{\partial}{\partial y}\left(k\frac{\partial}{\partial y}\right) + \frac{\Delta y^4}{24}\left[\frac{\partial}{\partial y}\left(k\frac{\partial^3}{\partial y^3}\right) + \frac{\partial^3}{\partial y^3}\left(k\frac{\partial}{\partial y}\right)\right] + 0(\Delta y^6), \quad (11)$$

$$\delta_x(k\delta_x)\delta_y(k\delta_y) = \Delta x^2\Delta y^2\frac{\partial}{\partial x}\left\{k\frac{\partial}{\partial y}\left(k\frac{\partial}{\partial y}\right)\right]\right\} + 0(\Delta x^3\Delta y^3). \quad (12)$$

Substituting in equation (2) and rearranging, the local accuracy can be written as:

$$\varepsilon_{i,j}^h = \Delta\tau^2\left\{\frac{1}{6}\left(\frac{\partial^3 t}{\partial\tau^3}\right)_{i,j}^h - \frac{1}{3C}\left[\frac{\partial}{\partial x}\left(k\frac{\partial}{\partial x}\left(\frac{\partial^2 t}{\partial\tau^2}\right)\right)_{i,j}^h + \frac{\partial}{\partial y}\left(k\frac{\partial}{\partial y}\left(\frac{\partial^2 t}{\partial\tau^2}\right)\right)_{i,j}^h\right] + \frac{4}{9}\frac{1}{C^2}\frac{\partial}{\partial x}\right.$$

$$\times \left[k\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(k\frac{\partial}{\partial y}\left(\frac{\partial t}{\partial \tau}\right)\right)\right)\right]^h_{i,j}\Bigg\}$$
$$-\frac{\Delta x^2}{24}\frac{1}{C}\left\{\left[\frac{\partial}{\partial x}\left(k\frac{\partial^3 t}{\partial x^3}\right)\right]^h_{i,j}+\left[\frac{\partial^3}{\partial x^3}\left(k\frac{\partial t}{\partial x}\right)\right]^h_{i,j}\right\}$$
$$-\frac{\Delta y^2}{24}\frac{1}{C}\left\{\left[\frac{\partial}{\partial y}\left(k\frac{\partial^3 t}{\partial y^3}\right)\right]^h_{i,j}+\left[\frac{\partial^3}{\partial y^3}\left(k\frac{\partial t}{\partial y}\right)\right]^h_{i,j}\right\}$$
$$+\ldots = 0(\Delta\tau^2 + \Delta x^2 + \Delta y^2) \qquad (13)$$

if it is assumed that the derivatives of t with respect to τ, x and y are continuous up to the order which is needed to define its principal part.

From formula (13) it follows immediately that:

$$\lim_{(\Delta\tau,\Delta x,\Delta y)\to 0} \varepsilon^h_{i,j} = 0 \qquad (14)$$

at each point (i,j,h) of the domain. The three-level scheme is therefore "consistent" with the differential equation.

The stability of formula (2) can be examined by writing it as the two-level system:

$$(1-X)(1-Y)T^{h+1}_{i,j} = (X+Y)T^h_{i,j} + (1+X)(1+Y)V^h_{i,j}$$
$$V^{h+1}_{i,j} = T^h_{i,j}$$

or, in vector form:

$$\begin{bmatrix}(1-X)(1-Y) & 0\\ 0 & 1\end{bmatrix}[W]^{h+1} = \begin{bmatrix}(X+Y) & (1+X)(1+Y)\\ 1 & 0\end{bmatrix}[W]^h$$

where:

$$X = \frac{2}{3}\frac{\Delta\tau}{C\,\Delta x^2}\,\delta_x(k\delta_x);\quad Y = \frac{2}{3}\frac{\Delta\tau}{C\,\Delta y^2}\,\delta_y(k\delta_y) \quad (15)$$

and:

$$[W] = [T, V]^T. \qquad (16)$$

If the two sides of the former matrix equation are premultiplied by the inverse of the square matrix on the left side, the result is:

$$[W_{i,j}]^{h+1} = \begin{bmatrix}\dfrac{X+Y}{(1-X)(1-Y)} & \dfrac{(1+X)(1+Y)}{(1-X)(1-Y)}\\ 1 & 0\end{bmatrix}[W_{i,j}]^h. \qquad (17)$$

The stability of the present scheme can then be investigated by using von Neumann's method locally at different parts of the domain under consideration, the coefficients being thought of as "frozen" to fixed values in each region [11].

Therefore it is assumed that the solution of the finite-difference system (17) can be represented by a Fourier expansion in which separation of space and time variables can be made. The general nth term of this expansion can be written as:

$$[\phi_n]\exp(I\gamma_x x)\exp(I\gamma_y y)$$

and since the analysis is applied to a region where k is considered constant formulae (15) can be rewritten as:

$$X \cong \frac{2}{3}\frac{k\,\Delta\tau}{C\,\Delta x^2}\,\delta_x^2;\qquad Y \cong \frac{2}{3}\frac{k\,\Delta\tau}{C\,\Delta y^2}\,\delta_y^2. \qquad (15')$$

Then using Euler's formulae the following relationships are derived:

$$X[\phi_n]\exp(Iy_x x)\exp(I\gamma_y y)$$
$$= [\phi_n]\exp(I\gamma_x x)\exp(I\gamma_y y)\times\left(-\frac{8}{3}\frac{k}{C}\frac{\Delta\tau}{\Delta x^2}\sin^2\frac{\gamma_x\Delta x}{2}\right)$$
$$Y[\phi_n]\exp(I\gamma_x x)\exp(I\gamma_y y). \qquad (15'')$$
$$= [\phi_n]\exp(I\gamma_x x)\exp(I\gamma_y y)\times\left(-\frac{8}{3}\frac{k}{C}\frac{\Delta\tau}{\Delta y^2}\sin^2\frac{\gamma_y\Delta y}{2}\right).$$

Substituting (15″) in equation (17) the result:

$$[\phi_n]^{h+1} = \begin{bmatrix}\dfrac{A+B}{(1-A)(1-B)} & \dfrac{(1+A)(1+B)}{(1-A)(1-B)}\\ 1 & 0\end{bmatrix}\times[\phi_n]^h \qquad (18)$$

is obtained, where:

$$A = -\frac{8}{3}\frac{k}{C}\frac{\Delta\tau}{\Delta x^2}\sin^2\frac{\gamma_x \Delta x}{2}$$

and:

$$B = -\frac{8}{3}\frac{k}{C}\frac{\Delta\tau}{\Delta y^2}\sin^2\frac{\gamma_y \Delta y}{2}.$$

The square matrix in equation (18) is called the amplification matrix of the system and its characteristic equation is:

$$(1 - A)(1 - B)\lambda^2 - (A + B)\lambda - (1 + A)(1 + B) = 0. \qquad (19)$$

In order that the computational procedure used becomes stable, von Neumann's condition requires that [11]:

$$\max|\lambda_m| \leqslant 1 \qquad (m = 1, 2)$$

where λ_m ($m = 1$, 2) are the eigenvalues of the amplification matrix (18) i.e. the roots of equation (19). In [7] it has been shown that the roots of the characteristic equation (19) satisfy von Neumann's condition for all values of $\Delta\tau$, Δx, Δy and of the thermophysical properties. This indicates unconditional stability.

The three level method is convergent too, since the Lax equivalence theorem states that [12]: "If the finite difference equations are consistent with the differential equation, then stability is the necessary and sufficient condition for convergence".

As the convergence of the scheme has been directly established by Lees in the one-dimensional case [9], the proof given here could appear unsatisfying. However it is not so, at least in so far as the possibility of application to situations different from those herein explicitly analyzed is concerned. The local order of accuracy of a formula can in fact be immediately determined, while if stability is usually hard to prove *a priori* it is always very easy to recognize experimentally.

INFLUENCE OF BOUNDARY CONDITIONS

Implicit reference has been so far made to boundary conditions of prescribed surface temperature. Boundary conditions of a different kind can however be treated in a formally analogous manner by suitably defining "equivalent thermal conductivities".

In the case of a surface with convective heat transfer coefficient specified, the heat flux:

$$q = \alpha\Delta L(T_e - T_{i,j})$$

can be decomposed into horizontal (q_x) and vertical components (q_y) which sum to q. With reference to Fig. 1b this is done by writing [14]:

$$q = q\sin^2\theta + q\cos^2\theta.$$

Then the following relationships are obtained:

$$q_x = q\sin^2\theta = \alpha\,\Delta L\sin^2\theta(T_e - T_{i,j}) = k_{x\,\mathrm{eq}}\frac{\Delta L}{\Delta x}\sin\theta(T_e - T_{i,j})$$

$$q_y = q\cos^2\theta = \alpha\,\Delta L\cos^2\theta(T_e - T_{i,j}) = k_{y\,\mathrm{eq}}\frac{\Delta L}{\Delta y}\cos\theta(T_e - T_{i,j})$$

from which the equivalent thermal conductivities are derived:

$$k_{x\,\mathrm{eq}} = \alpha\,\Delta x\sin\theta \qquad (20')$$

and

$$k_{y\,\mathrm{eq}} = \alpha\,\Delta y\cos\theta. \qquad (20'')$$

The same procedure can be followed in the case of a surface with heat flux specified, by assigning the external points a "false" temperature of $+1{\cdot}0$°C for correct coefficient manipulation [8]. Particularly, if the surface is adiabatic the result is:

$$k_{x\,\mathrm{eq}} = k_{y\,\mathrm{eq}} = 0. \qquad (21)$$

The influence of boundary conditions which are different from those of the first kind on the stability of a finite difference method has been discussed elsewhere [7] and therefore is not covered again here. It must be pointed out however that the three-time level method remains unconditionally stable, even with boundary conditions of the second and third kind.

RESULTS

The numerical method described in the previous sections has been coded in Fortran IV, obtaining a program to determine two-dimensional thermal fields in the square cross-section of an infinite prism with temperature dependent thermal properties, placed in a fluid which produces a constant and uniform convection coefficient over the external surface. The symmetry of the problem allows the simplification of considering only one quarter of the cross-section, with two boundaries exchanging heat by convection while the others are kept insulated.

The program has first been tested in the particular case of constant thermophysical properties and a unit step in the fluid temperature. The temperature at the centre, at the surface mid-side and at the corner surface of the sample, computed for different values of the convection coefficient by using a grid of 11×11 nodal points, have been compared with the analytical solution [10]. The accuracy of the computations is always better than 1 per cent [4].

When the thermophysical properties depend on temperature, no analytical solution exists and therefore the program can be checked only against the results of experiments.

Several heating and cooling tests in air have then been carried out on samples with a square cross-section, thermally insulated at the ends in order to ensure two-dimensional heat conduction at the mid-length [4]. The test substance used in the samples was "Tylose", a water and methylcellulose (77 per cent and 23 per cent in weight) mixture whose thermophysical properties are strongly temperature dependent [3, 13]. The experimental apparatus and procedures are described in [4].

The centre and the surface temperatures of the samples have been recorded and then, with reference to the known thermophysical properties and linear dimensions of the samples and to the behaviour of the air temperature, nodes temperatures have been computed. The convective heat transfer coefficients have not been measured but the values used gave the best fit between experimental and computed results.

The experimental and computed curves are reported in [4]: the difference between experimental results and the results yielded by the three-level method is less than 1 K with temperature variations during tests in terms of 50 K and over.

Run time for the Fortran program is less than two minutes on a CDC 6600 computer with a grid of 11×11 nodal points and a maximum number of 1000 time steps.

CONCLUSIONS

The finite-difference method presented in this paper has enabled to deal with multidimensional heat conduction problems when the thermophysical properties depend on temperature. The computer procedure has been found to be reasonably facile and accurate.

As a final remark it is worth indicating that the extension of this work to more complex geometrical configurations would require only the additional effort of writing a computer program flexible in region description, on the basis, for example, of the excellent ones already developed for standard A.D.I. methods [8].

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SOLUTION DES EQUATIONS NON LINEAIRES DE CONDUCTION THERMIQUE PAR DES METHODES NUMERIQUES

Résumé— On présente un schéma implicite à trois niveaux dans le temps pour la résolution numérique des équations de la conduction de la chaleur quand les propriétés thermophysiques dépendent de la température. Comme ces dernières sont évaluées à l'époque intermédiaire, la compléxité de la résolution du système d'équations non-linéaires à chaque pas dans le temps est évitée. Dans le cas des conditions aux limites de première espèce, on montre que la méthode est absolument stable et convergente. Les conditions aux limites de seconde et troisième espèce sont traitées de la même manière à l'aide de "conductivités thermiques équivalentes" convenablement définies.

Cette procédure est utilisée pour simuler plusieurs essais de chauffage et de refroidissement par convection d'échantillons de "Tylose" qui ont des propriétés thermophysiques dépendant fortement de la température. Des champs de température dans des carrés soumis aux conditions aux limites de seconde et troisième espèce sont alors calculés et comparés de manière satisfaisante aux résultats expérimentaux.

ZUR LÖSUNG NICHTLINEARER WÄRMELEITUNGSGLEICHUNGEN DURCH NUMERISCHE METHODEN

Zusammenfassung—Ein implizites Differenzenschema mit drei Zeitebenen zur numerischen Lösung mehrdimensionaler Wärmeleitungsgleichungen mit temperaturabhängigen thermophysikalischen Eigenschaften wird beschrieben. Da die thermoplastischen Eigenschaften bei der mittleren Zeitebene genommen werden, wird die Schwierigkeit umgangen, bei jedem Zeitschritt einen Satz nichtlinearer Gleichungen lösen zu müssen. Für die Randbedingungen erster Art zeigt es sich, dass die Methode in jedem Fall stabil und konvergent ist. Randbedingungen zweiter und dritter Art werden ähnlich behandelt, indem in geeigneter Weise "äquivalente Wärmeleitfähigkeiten" definiert werden.

Dieses Verfahren wird angewendet, um verschiedene konvektive Aufheiz- und Kühlvorgänge an "Tylose"-Proben zu simulieren, die stark temperaturabhängige thermophysikalische Eigenschaften haben. Temperaturfelder in Vierkantprismen bei Randbedingungen zweiter und dritter Art werden auf diese Weise berechnet. Sie stimmen in befriedigender Weise mit experimentellen Ergebnissen überein.

О РЕШЕНИИ НЕЛИНЕЙНЫХ УРАВНЕНИЙ ТЕПЛОПРОВОДНОСТИ ЧИСЛЕННЫМИ МЕТОДАМИ

Аннотация—Описывается трехслойная неявная схема численного решения многомерных уравнений теплопроводности, когда теплофизические свойства зависят от температуры. Поскольку теплофизические свойства рассчитываются для промежуточного слоя, это позволяет избежать затруднений при решении системы нелинейных уравнений на каждом шаге по времени. Показано, что в случае граничных условий первого рода этот метод дает устойчивую сходимость результатов. Затем, определив

соответствующим образом «эквивалентную теплопроводность», рассматривают граничные условия второго и третьего рода.

Это метод использовался для моделирования конвективного нагрева на образцах «тилозы», теплофизические свойства которой сильно зависят от температуры.

Таким образом рассчитаны температурные поля в телах квадратной формы при граничных условиях второго и третьего рода; сравнение с экспериментом показывает удовлетворительное совпадение.